Diagram Approach to the Theory of Collisionless Plasma Turbulence

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From the Vlasov equation written as a nonlinear equation for the distribution function f, a perturbation expansion of the fluctuating part of f, $\tilde{f}=f-\langle f\rangle$, is carried out in terms of the linear solution using a diagram representation similar to that of Wyld. From this a diagram expansion is obtained for the correlation function $U=\langle\tilde{f}\,\tilde{f}\rangle$ and similar diagram series for two auxiliary functions, the averaged linear response function $G=\langle\hat{G}\rangle$ and the generalized vertex function Γ. Approximate asymptotic equations for U imply the summation of appropriate infinite subsets of diagrams. To derive such approximations two criteria are applied: 1) Consistency with the basic conservation laws (particle number, momentum, energy); 2) Selection of diagrams analogous to those leading to good approximations in Kraichnan's random oscillator problem, the principal results of which are regained. Contrary to Wyld's summation procedure diagram summation are not carried out directly in the expansion of U, but in the expansions of the functions $\langle\tilde{f}\,\tilde{f}\,\tilde{f}\rangle$ and $\langle\tilde{f}\,\hat{G}\rangle$, which appear on the right hand side of the differential equations for U and G. This automatically guaranties the validity of the conservation laws. The first approximation we discuss, which is equivalent to the quasi-Gaussian approximation well known from hydrodynamic turbulence, leads to the kinetic wave equation of weak turbulence. The next approximation consists of a coupled set of equations for U and G which are identical with the equations of the symmetrical random coupling model of Orszag and Kraichnan. The third approximation derived implies the generalized vertex function Γ. The first and the third of these approximations cannot be obtained in Wyld's summation scheme. On the other hand it does not seem possible to derive the third approximation for the nonlinear Vlasov equation using Kraichnan's diagram technic.

Hydrodynamic and hydromagnetic turbulence, especially in incompressible media, have been investigated for a rather long time. A collisionless plasma however can, in general, not be treated in the hydrodynamic approximation but requires a microscopic description. In the last decade various microscopic approaches have been discussed, the simplest being the quasi-linear theory [1–3]. Most of these approximations, however, are restricted to weak turbulence. They treat a plasma as a system of nearly independent weakly interacting particles and waves. Such an approach implies that the number of particles interacting strongly with a wave (resonant particles) is small which limits the oscillation amplitude. To treat these particles correctly, in particular to take into account the effect of reflection by the potential hump of a wave (generally called particle trapping) which becomes increasingly important for stronger turbulence, it is necessary to incorporate the effect of the fluctuating fields on the particle trajectories from the beginning instead of starting with free particle propagation. This leads to coupled equations for the particle motion and the development of field fluctuations. In recent time some attempts have been made in this direction. Dupree [4] directly uses the physical picture of statistical particle orbits, while Orszag and Kraichnan [5] introduce the averaged response function of the exact system to an infinitesimal disturbance which has no immediate interpretation in terms of single particle motion.

There are essentially two approaches to derive closed statistical equations for collisionless plasma turbulence.

(a) The Vlasov equation is treated as a linear equation for the distribution function f, the electric field E being statistically prescribed by the set of moments $\langle E\rangle$, $\langle EE\rangle, \ldots$. Some approximate closed equations for the moments $\langle f\rangle$, $\langle ff\rangle, \ldots$ are derived and then the moments of E are related to those of f by Poisson's equation. The pseudolinear model of Orszag and Kraichnan [5] is a special approximation of this approach, and Dupree's [4] theory is based on similar ideas.

(b) The Vlasov equation is treated as a nonlinear equation for the distribution function f. This seems

[1] W. E. Drummond and D. Pines, Nucl. Fusion Suppl. 1962, Part 3, 1049.

[2] A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Nucl. Fusion **1**, 82 [1961].

[3] See e. g. B. B. Kadomtsev, Plasma Turbulence, Academic Press, Londond and New York 1965.

[4] T. H. Dupree, Phys. Fluids **9**, 1773 [1966].

[5] S. A. Orszag and R. H. Kraichnan, Phys. Fluids **10**, 1720 [1967].

to be conceptionally simpler, since only one set of moments, $\langle f \rangle$, $\langle f f \rangle, \ldots$ appears. For instance, the selfconsistent Vlasov models of ORSZAG and KRAICHNAN[5] are along this line. In the present paper we are concerned with this approach.

It is often convenient (and has become fashionable in many branches of physics) to use diagram technics to derive approximate theories from fundamental equations. In hydrodynamic turbulence two different diagram approaches have been introduced. WYLD's[6] diagrams are related to those extensively used in field theory and quantum statistics, while KRAICHNAN's[7] are less comprehensive, since they do not give a direct representation of the various perturbation expansion terms but of combinations of phase factors artificially introduced into the dynamical equations. MIKHAILOVSKY[8] has applied Wyld's diagram method to the linear Vlasov problem (a). The analysis is however much more cumbersome than in the hydrodynamic case treated by Wyld, involving an infinite number of equations for infinitely many functions. In this paper we shall use a generalization of Wyld's diagrams to consider the selfconsistent Vlasov equation (b). We shall carry out a perturbation expansion of the fluctuating part of the distribution function f in terms of the solution of the linearized equation and from this obtain a perturbation expansion of the correlation function $U = \langle \tilde{f} \tilde{f} \rangle$, using an assumption about the initial statistics. The method is not restricted to stationary and homogeneous systems which was required in the paper of Wyld[6], LEE[9], Mikhailovsky[8].

The general summation procedure of Wyld, however, which results in a set of three coupled infinite power series integral equations for the correlation function, the propagator and the vertex function, seems to be of little use. Breaking these consolidated series at some arbitrary point will in most cases not give a good approximation, as can clearly be shown in simple examples. Consideration of sufficiently simple model problems can be a great help in deciding what classes of diagrams in the real problem should be summed to give meaningful approximate equations. We therefore shall briefly repeat the results of KRAICHNAN's[7] random oscillator model translating them into the simpler and more comprehensive language of Wyld's diagrams. The analogy to this simple linear problem is however not sufficient to determine physically meaningful approximations for the more complicated nonlinear Vlasov problem. It is therefore important to insure that certain conservation laws such as energy conservation, that are valid in the exact dynamical system, are also satisfied by the approximate statistical equations. The integral equations that arise from a truncation of the consolidated series of WYLD[6] and LEE[9] are in general not energy conserving. We therefore adopt a different procedure. We will not carry out summations in the perturbation expansion of the correlation function $U = \langle \tilde{f} \tilde{f} \rangle$, but consider the differential equation for $\langle \tilde{f} \tilde{f} \rangle$ which involves the third order correlation function $\langle \tilde{f} \tilde{f} \tilde{f} \rangle$ and carry out partial summations in the expansion of the latter function. We derive and discuss several approximate equations which imply diagram summations analogous to "good" approximations of the random oscillator problem and which satisfy the dynamical conservation laws. The first nontrivial approximation is equivalent to the quasi Gaussian moment's closure (well known from hydrodynamic turbulence), which leads to the weak turbulence equations[3]. In the next approximation a generalized propagator G is introduced. The set of equations for U and G is identical to the random coupling model of ORSZAG and KRAICHNAN[5]. A (hopefully) more refined approximation then is obtained by taking into account vertex corrections in an appropriate way, leading to a set of equations which have not been derived previously.

I. Diagram Representation of the Perturbation Expansion

We consider the self-consistent Vlasov equation for a one-component plasma with neutralizing charge background without magnetic field in the electrostatic approximation (these restrictions are only for convenience, not imposed by the method):

$$\frac{\partial f(\boldsymbol{x},\boldsymbol{v},t)}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f(\boldsymbol{x},\boldsymbol{v},t)}{\partial \boldsymbol{x}} + \boldsymbol{C}(\boldsymbol{x})\,[f(\boldsymbol{x},\boldsymbol{v},t)] \cdot \frac{\partial f(\boldsymbol{x},\boldsymbol{v},t)}{\partial \boldsymbol{v}} = 0\,, \tag{1}$$

$$\boldsymbol{C}(\boldsymbol{x})\, f(\boldsymbol{x},\boldsymbol{v},t) \equiv \int \boldsymbol{C}(\boldsymbol{x},\boldsymbol{x}')\, f(\boldsymbol{x}',\boldsymbol{v}',t)\, \mathrm{d}\boldsymbol{x}'\, \mathrm{d}\boldsymbol{v}' = \frac{e}{m} \boldsymbol{E}(\boldsymbol{x},t)\,.$$

[5] ...

[6] H. W. WYLD, JR., Ann. Phys. New York **14**, 143 [1961].

[7] R. H. KRAICHNAN, J. Math. Phys. **2**, 124 [1961].

[8] A. B. MIKHAILOWSKY, Nucl. Fusion **4**, 321 [1964].

[9] L. L. LEE, Ann. Phys. New York **32**, 292 [1965].

Introducing averages over an ensemble of equivalent systems and using the notation

$$f(\boldsymbol{x},\boldsymbol{v},t)=F(\boldsymbol{x},\boldsymbol{v},t)+\tilde{f}(\boldsymbol{x},\boldsymbol{v},t),\; F(\boldsymbol{x},\boldsymbol{v},t)\equiv\langle f(\boldsymbol{x},\boldsymbol{v},t)\rangle,$$

$$\boldsymbol{C}(\boldsymbol{x})\,F(\boldsymbol{x},\boldsymbol{v},t)=\frac{e}{m}\langle\boldsymbol{E}(\boldsymbol{x},t)\rangle,\qquad \boldsymbol{C}(\boldsymbol{x})\,\tilde{f}(\boldsymbol{x},\boldsymbol{v},t)=\frac{e}{m}\tilde{\boldsymbol{E}}(\boldsymbol{x},t) \tag{2}$$

we obtain from (1)

$$\frac{\partial F(\boldsymbol{x},\boldsymbol{v},t)}{\partial t}+\boldsymbol{v}\cdot\frac{\partial F}{\partial\boldsymbol{x}}+\frac{e}{m}\langle\boldsymbol{E}(\boldsymbol{x},t)\rangle\cdot\frac{\partial F}{\partial\boldsymbol{v}}=-\frac{e}{m}\Big\langle\tilde{\boldsymbol{E}}(\boldsymbol{x},t)\frac{\partial\tilde{f}}{\partial\boldsymbol{v}}(\boldsymbol{x},\boldsymbol{v},t)\Big\rangle, \tag{3}$$

$$\frac{\partial\tilde{f}(\boldsymbol{x},\boldsymbol{v},t)}{\partial t}+\boldsymbol{v}\cdot\frac{\partial\tilde{f}}{\partial\boldsymbol{x}}+\frac{e}{m}\langle\boldsymbol{E}\rangle\cdot\frac{\partial\tilde{f}}{\partial\boldsymbol{v}}+\frac{\partial F}{\partial\boldsymbol{v}}\cdot\boldsymbol{C}(\boldsymbol{x})\,\tilde{f}=-\boldsymbol{C}(\boldsymbol{x})\,\tilde{f}(\boldsymbol{x},\boldsymbol{v},t)\cdot\frac{\partial\tilde{f}(\boldsymbol{x},\boldsymbol{v},t)}{\partial v}+\Big\langle\boldsymbol{C}(\boldsymbol{x})\,\tilde{f}(\boldsymbol{x},\boldsymbol{v},t)\cdot\frac{\partial\tilde{f}(\boldsymbol{x},\boldsymbol{v},t)}{\partial v}\Big\rangle. \tag{4}$$

The last equation can be written in the following form, suppressing the arguments $\boldsymbol{x}$, $\boldsymbol{v}$,

$$L_t\,\tilde{f}(t)=\boldsymbol{C}\,\tilde{f}(t)\cdot\boldsymbol{D}\,\tilde{f}(t)-\langle\boldsymbol{C}\,\tilde{f}(t)\cdot\boldsymbol{D}\,\tilde{f}(t)\rangle,\quad \boldsymbol{D}\equiv-\frac{\partial}{\partial\boldsymbol{v}}. \tag{5}$$

Introducing $G^0(t\,|\,t')\equiv G^0(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}',\boldsymbol{v}',t')$ the Green's function of the linear part of (4),

$$L_t\,G^0(t\,|\,t')=0,\qquad G^0(t\,|\,t')=0,\qquad t<t',$$
$$G^0(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}',\boldsymbol{v}',t)=\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(\boldsymbol{v}-\boldsymbol{v}'), \tag{6}$$

the formal solution of (5) is

$$\tilde{f}(t)=\tilde{f}^0(t)+G^0(t)\,[\boldsymbol{C}\,\tilde{f}(t)\cdot\boldsymbol{D}\,\tilde{f}(t)-\langle\boldsymbol{C}\,\tilde{f}(t)\cdot\boldsymbol{D}\,\tilde{f}(t)\rangle],$$
$$\tilde{f}^0(t)\equiv\int\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{v}'\,G^0(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}',\boldsymbol{v}',0)\,\tilde{f}(\boldsymbol{x}',\boldsymbol{v}',0), \tag{7}$$
$$G^0(t)\,A(t)\equiv\int_0^t\mathrm{d}t'\int\mathrm{d}\boldsymbol{x}'\,\mathrm{d}\boldsymbol{v}'\,G^0(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}',\boldsymbol{v}',t')\,A(\boldsymbol{x}',\boldsymbol{v}',t').$$

We shall call G^0 the elementary propagator. It should be noted that L_t contains $F(t)$ which is determined by Eq. (3). Therefore the linearized part of (5) constitutes in general an additional simultaneous equation.

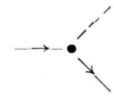
We now introduce a diagram representation similar to that of Wyld and define the following diagram elements:

$$- - -\qquad \tilde{f}^0(X),\qquad X\equiv\{\boldsymbol{x},\boldsymbol{v},t\} \tag{8}$$
$$\longrightarrow\qquad G^0(X\,|\,X') \tag{9}$$
$$\bullet\qquad \Gamma^0(X\,|\,X',X'')\equiv\boldsymbol{C}(\boldsymbol{x},\boldsymbol{x}')\,\delta(\boldsymbol{x}-\boldsymbol{x}'')\,\delta'(\boldsymbol{v}-\boldsymbol{v}'')\,\delta(t-t')\,\delta(t-t'') \tag{10}$$
$$\circ\qquad \Gamma_s^0(X\,|\,X',X'')\equiv\Gamma^0(X\,|\,X',X'')+\Gamma^0(X\,|\,X'',X'). \tag{11}$$

We call Γ^0 the elementary vertex. In a diagram three lines join together in a vertex. There is always one ingoing G^0-line; and two lines leave, either two G^0-lines or one G^0- and one $\tilde{f}^0$-line or two $\tilde{f}^0$-lines. A vertex inserted into a diagram implies integration over X', X'':

$$\int\mathrm{d}X'\,\mathrm{d}X''\,\Gamma^0(X\,|\,X',X'')\,A(X')\,B(X'')\equiv\boldsymbol{C}\,A\cdot\boldsymbol{D}\,B,$$

a G^0-line inserted into a diagram integration over X'.

We use the convention that in  the line leaving upward is operated on by $\boldsymbol{C}$, the other line by $\boldsymbol{D}$. Since Γ^0 is not symmetric here, it is convenient to introduce the symmetrized vertex Γ_s^0, Eq. (11).

Contrary to the homogeneous hydrodynamic case treated by Wyld, the averaged quantity on the right hand side of (4) does in general not vanish. Therefore, in order to carry out a perturbation expansion of (4), we need some statistical assumption equivalent to the maximal randomness assumption about thet stirring forces introduced by Wyld. Since we shall not restrict our discussion to stationary systems, the most natural way is to require a similar property for the initial state of the system. The simplest assumption is to take $\tilde{f}(\boldsymbol{x},\boldsymbol{v},0)$ to have a multivariate Gaussian distribution, i. e. every odd order moment vanishes and every even order moment splits into a sum of products of second order moments:

$$\langle\tilde{f}(\boldsymbol{x},\boldsymbol{v},0)\,\tilde{f}(\boldsymbol{x}',\boldsymbol{v}',0)\rangle=U(\boldsymbol{x},\boldsymbol{v},\boldsymbol{x}',\boldsymbol{v}',0),$$
$$\langle\tilde{f}(\boldsymbol{x},\boldsymbol{v},0)\,\tilde{f}(\boldsymbol{x}',\boldsymbol{v}',0)\,\tilde{f}(\boldsymbol{x}'',\boldsymbol{v}'',0)\rangle=0,$$
$$\langle\tilde{f}(\boldsymbol{x},\boldsymbol{v},0)\,\tilde{f}(\boldsymbol{x}',\boldsymbol{v}',0)\,\tilde{f}(\boldsymbol{x}'',\boldsymbol{v}'',0)\,\tilde{f}(\boldsymbol{x}''',\boldsymbol{v}''',0)\rangle$$
$$=U(\boldsymbol{x},\boldsymbol{v},\boldsymbol{x}',\boldsymbol{v}',0)\,U(\boldsymbol{x}'',\boldsymbol{v}'',\boldsymbol{x}''',\boldsymbol{v}''',0)+U(\boldsymbol{x},\boldsymbol{v},\boldsymbol{x}'',\boldsymbol{v}'',0)\,U(\boldsymbol{x}',\boldsymbol{v}',\boldsymbol{x}''',\boldsymbol{v}''',0)+U(\boldsymbol{x},\boldsymbol{v},\boldsymbol{x}''',\boldsymbol{v}''',0)\,U(\boldsymbol{x}',\boldsymbol{v}',\boldsymbol{x}'',\boldsymbol{v}'',0)\ \text{etc.} \tag{12}$$

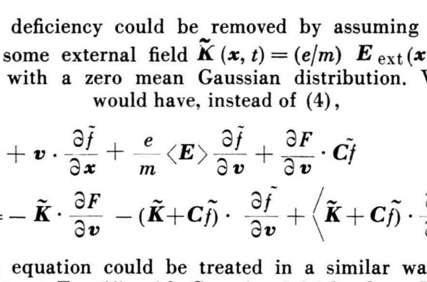
This seems very appropriate if the initial state is characterized by a low fluctuation level from which turbulence is generated by some instability. It should however be mentioned that this choice has an unaesthetic feature. Strictly speaking a Gaussian assumption for $\tilde{f}(\boldsymbol{x}, \boldsymbol{v})$ is incompatible with the positivity of $f(\boldsymbol{x}, \boldsymbol{v})$ in every realization, since a Gaussian distribution extents to infinity and there are large negative $\tilde{f}(\boldsymbol{x}, \boldsymbol{v})$ present making the total distribution function negative *. For $U(0) \ll F(0)^2$ these forbidden realizations have a negligible weight. Nevertheless this can perhaps entail a catastrophic evolution by some instability. We will come back to this point later when discussing briefly the behavior of the averaged distribution function. In the paper of ORSZAG and KRAICHNAN [5] the requirement of Gaussian initial distribution is not mentioned but for the derivation of the model equations the reader is referred back to KRAICHNAN's original paper [7], where this requirement was explicitely stated.

Using the symbols (8) – (11) and the statistical properties of $\tilde{f}^0(t)$ which are the same as for $\tilde{f}(0)$ in (12) and denoting U^0 by a wavy line,

$$\sim\!\sim\!\sim \quad \langle \tilde{f}^0(\boldsymbol{x}, \boldsymbol{v}, t)\, \tilde{f}^0(\boldsymbol{x}', \boldsymbol{v}', t')\rangle = U^0(\boldsymbol{x}, \boldsymbol{v}, t, \boldsymbol{x}', \boldsymbol{v}', t'), \tag{13}$$

we can write down in diagram language the perturbation expansion of f in terms of f^0 obtained by iterating Eq. (7)

$$\tilde{f}(\boldsymbol{x}, \boldsymbol{v}, t) = \tag{14}$$



* This deficiency could be removed by assuming $f(0) = 0$ and some external field $\tilde{\boldsymbol{K}}(\boldsymbol{x}, t) = (e/m)\, \boldsymbol{E}_{\text{ext}}(\boldsymbol{x}, t)$ present with a zero mean Gaussian distribution. We then would have, instead of (4),

$$\frac{\partial \tilde{f}}{\partial t} + \boldsymbol{v} \cdot \frac{\partial \tilde{f}}{\partial \boldsymbol{x}} + \frac{e}{m} \langle \boldsymbol{E} \rangle \frac{\partial \tilde{f}}{\partial \boldsymbol{v}} + \frac{\partial F}{\partial \boldsymbol{v}} \cdot \boldsymbol{C}\tilde{f}$$
$$= -\tilde{\boldsymbol{K}} \cdot \frac{\partial F}{\partial \boldsymbol{v}} - (\tilde{\boldsymbol{K}} + \boldsymbol{C}\tilde{f}) \cdot \frac{\partial \tilde{f}}{\partial \boldsymbol{v}} + \left\langle (\tilde{\boldsymbol{K}} + \boldsymbol{C}\tilde{f}) \cdot \frac{\partial \tilde{f}}{\partial \boldsymbol{v}} \right\rangle .$$

This equation could be treated in a similar way as we will treat Eq. (4) with Gaussian initial values. The final equations would be essentially the same if the magnitude and the duration time of $\tilde{\boldsymbol{K}}$ are made sufficienty small.

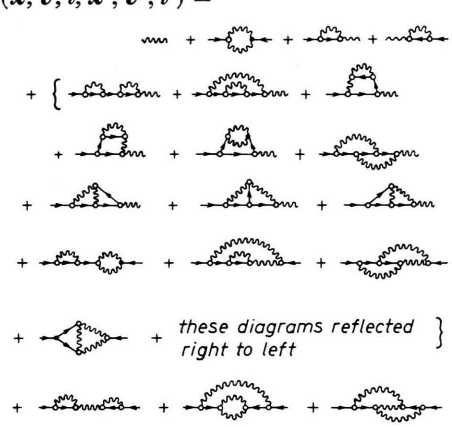
Averaging in each order leads to diagrams with parts that are connected with the rest by a single G^0-line and which we call closed-tree like. Their appearance considerably enlarges the zoology of different $\tilde{f}$-diagrams compared to Wyld's case.

The quantity of most interest is the correlation function

$$U(\boldsymbol{x}, \boldsymbol{v}, t, \boldsymbol{x}', \boldsymbol{v}', t') = \langle \tilde{f}(\boldsymbol{x}, \boldsymbol{v}, t)\, \tilde{f}(\boldsymbol{x}', \boldsymbol{v}', t') \rangle$$

which for instance permits calculation of the fluctuating field intensity $\langle \tilde{E}(\boldsymbol{x}, t)^2 \rangle$, and the time evolution of $F(t)$ by Eq. (3). We construct an infinite set of diagrams for U in terms of U^0 by multiplying $\tilde{f}(\boldsymbol{x}, \boldsymbol{v}, t)$ by $\tilde{f}(\boldsymbol{x}', \boldsymbol{v}', t')$ both represented by their diagram expansion (14) and averaging:

$$U(\boldsymbol{x}, \boldsymbol{v}, t, \boldsymbol{x}', \boldsymbol{v}', t') =$$



$$+ \ldots \tag{15}$$

The general structure of the U-diagrams is the same as in Wyld's hydrodynamic case. The appearance of a symmetrized vertex $\Gamma_s{}^0$ corresponds to a factor of 2 in Wyld s diagram expansion. It is interesting to note that no closed-tree like diagrams appear in (15). This is valid in any perturbation order, the prove will be given in the appendix.

II. A Simple Example: Kraichnan's Random Oscillator

Since the nonlinear behaviour (mode interaction) is a basic feature of turbulence, the nonlinear part of Eq. (4) cannot be considered small; a finite order perturbation solution in terms of $\tilde{f}^0$ is invalidated by secular effects after a very short time, which is of little interest. To obtain a meaningful long-time equation one therefore has to sum an appropriate infinite subset of perturbation terms. To carry out special infinite summations is the main purpose of Wyld's paper [6]. In analogy to quantum statistics Wyld introduces, in addition to the correlation function U, two auxiliary functions, the generalized propagator G and the generalized vertex Γ. These are defined by Wyld as sums of special types of diagrams, which are parts of U-diagrams. Wyld's summations can be described as follows: Select all U, G, Γ-diagrams which do not contain as genuine parts other U, G, Γ-diagrams, i. e. which are irreducible with respect to U, G, Γ-parts, and replace all elementary quanities U^0, G^0, Γ^0 by the exact ones, U, G, Γ. This leaves us with three coupled equations of the form of infinite power series. These so-called consolidated expansions must be closed at some point to yield finite equations. Any approximation has the form of an asymptotic equation, and it seems plausible that taking into account higher order irreducible diagrams would yield better quantitative agreement. This is however by no means true. The summation of an infinite number of perturbation terms does not automatically provide a finite asymptotic result, as was clearly shown by KRAICHNAN [7]. Kraichnan investigated the simple but not trivial model of an ensemble of independent harmonic oscillators with a random frequency, which is a dynamically linear but stochastically nonlinear problem with the same basic structure as the linear Vlasov equation with an externally prescribed random electric field [s. introduction, approach (a)]. We want to restate the principle results of this model using the language of Wyld's diagrams. The basic equation is

$$\frac{dx(t)}{dt} + i\,\omega\,x(t) = 0\,. \qquad (16)$$

The quantity of most interest in this linear problem is the averaged Green's function or propagator $G(t) = \langle \hat{G}(t) \rangle$, where $\hat{G}$ is defined by

$$x(t) = \hat{G}(t)\,x(0) \quad \text{and} \quad \langle x(t) \rangle = G(t)\,\langle x(0) \rangle$$

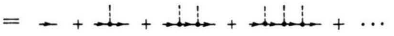
(we need not consider the correlation function $U = \langle \tilde{x}\tilde{x} \rangle$, $\tilde{x} = x - \langle x \rangle$, since in the linear equation (16) G is independent of U. A similar treatment of U is possible). The equation for $\hat{G}$ is:

$$\frac{d\hat{G}(t)}{dt} + i\,\omega\,\hat{G}(t) = 0, \quad \hat{G}(0) = 1, \quad \hat{G}(t) = 0, \quad t < 0, \qquad (17)$$

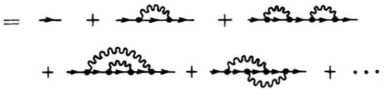
or after Laplace transformation

$$\hat{G}(p) = 1/p + (1/i\,p)\,\omega\,\hat{G}(p)\,. \qquad (18)$$

Introducing diagram elements analogous to (8) to (10), i. e. $\omega \to \tilde{f}^0$, $1/p \to G^0$, $1/i \to \Gamma^0$, the iteration solution of (18) can be written as:

$G(p) =$  (19)

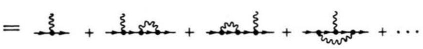
Averaging (19) and assuming ω to have a Gaussian distribution we obtain (with $\langle \omega^2 \rangle = \omega_0^{\,2}$ represented by a wavy line)

$G(p) =$  (20)

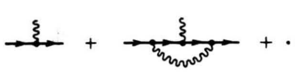
which is the iteration solution of the equation

$$G(p) = 1/p + (1/i\,p)\,\langle \omega\,\hat{G}(p) \rangle\,. \qquad (21)$$

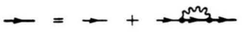
In view of the treatment of the Vlasov equation in section III we do not perform summations in the expansion of G, Eq. (20), which would correspond to Wyld's approach, but in the expansion of the function $\langle \omega\,\hat{G} \rangle$ which is the essential quantity on the right hand side of (21),

$\langle \omega\,\hat{G} \rangle =$  (22)

The simplest infinite partial summations possible in (20) or (22) are equivalent to breaking the moments hierarchy obtained from (18) at some finite order (cumulant discard method). This provides finite asymptotic results which however have the deficiency of infinite correlation time in contrast to the exact solution. A qualitative improvement over the moments' method is obtained by a refined kind of summation. Take all diagrams of (22) that are irreducible with respect to G-parts and replace all G^0-lines by G-lines (thick straigh lines), which gives the consolidated expansion of G with respect to G-parts:

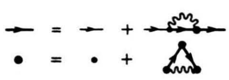
$\langle \omega\,\hat{G}(p) \rangle =$  (23)

Breaking this expansion at the lowest order and inserting $\langle \omega\,\tilde{G} \rangle$ into (21) we obtain the equation

 (24)

which gives rather good agreement with the exact result. This approximation is called by Kraichnan the random coupling model by reasons apparent from his paper [7]. It has the exceptional property that it represents the exact solution of a statistical model which guarantees that certain realizability conditions are satisfied.

Inclusion of a finite number of higher order terms in (23) will not only not improve the lowest order result but give a highly unphysical behaviour comparable to simple finite order truncations of the perturbation series (19). However a considerable improvement is found by summing a special infinite class of irreducible diagrams in (23), which can be seen to be the lowest order nontrivial truncation of a new consolidated series including a generalized vertex function $\Gamma(p)$. This approximation consists of two coupled equations for G and Γ (we denote Γ by a solid circle):


(25)

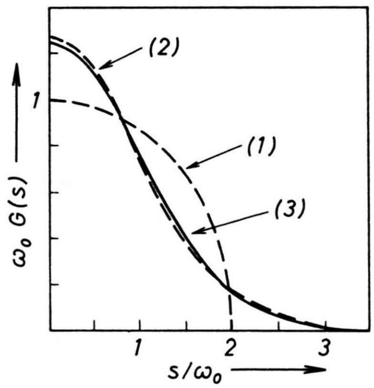


Fig. 1. Comparison of $G(s) = \text{Re}\, G(p)|_{p=-is}$ for the random coupling approximation (1) and the vertex approximation (2) with the exact solution (3) from KRAICHNAN [7].

Higher order truncations of the vertex expansion will again show a very unphysical behaviour. These results suggest that to obtain the next meaningful approximation one has again to sum a special infinite class of irreducible diagrams in the vertex expansion which is associated with the introduction of a new function with four terminals. It seems however not to be possible to write this further approximation in the form of equations like (24) or (25).

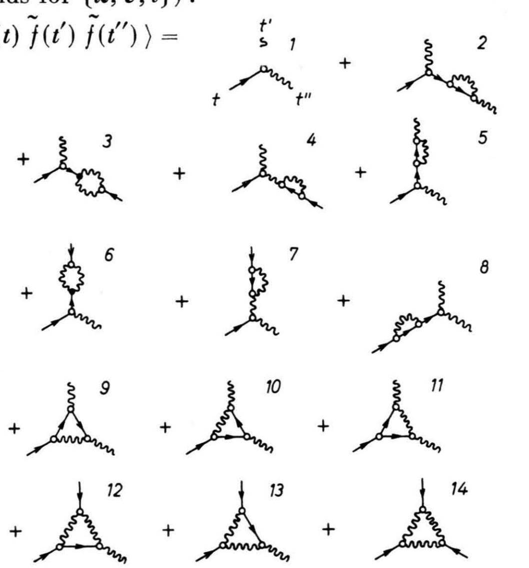
III. Approximations for the Selfconsistent Vlasov Problem

We have given a rather detailed presentation of the oscillator model because it is so transparent that it not only has a simple exact solution but also allows analytical treatment of various approximations (the perhaps simplest nonlinear model, the three wave model (KRAICHNAN [10]), requires extensive numerical computation already for the random coupling approximation). On the other hand the generalization of the results of the linear oscillator model to nonlinear problems is not unique. It is therefore important when constructing approximations for the nonlinear Vlasov problem to have further criteria in addition to the analogy with the oscillator model. Such criteria are provided by the conservation laws obeyed by the exact equations, such as particle number, momentum, and energy conservation. Since conservation laws are intimately connected with differential equations, it is not easy to extract energy conserving approximations from Wyld's summation scheme which leads to integral equations, and indeed most of these approximations do violate the conservation laws as was observed by KRAICHNAN [11]. We therefore shall not apply Wyld's method summing up terms in the iteration solution of U, Eq. (15), but consider the differential equation for U obtained by multiplying Eq. (4) by $f(\boldsymbol{x}', \boldsymbol{v}', t')$ and averaging (using the notation of Eq. (5)):

$$L_t\, U(t,t') = \langle \boldsymbol{C}\, \tilde{f}(t) \cdot \boldsymbol{D}\, \tilde{f}(t)\, \tilde{f}(t') \rangle. \qquad (26)$$

On the right hand side of (26) the third order moment $\langle \tilde{f}\tilde{f}\tilde{f} \rangle$ appears. Replacing each factor $\tilde{f}$ by its expansion (14) we obtain the perturbation expansion of this function (as before argument t stands for $\{\boldsymbol{x}, \boldsymbol{v}, t\}$):

$$\langle \tilde{f}(t)\, \tilde{f}(t')\, \tilde{f}(t'') \rangle =$$



+ diagrams 1 – 13 rotated through 120° and 240°
+ . . . . (27)

[10] R. H. KRAICHNAN, Phys. Fluids 6, 1603 [1963].

[11] R. H. KRAICHNAN, in "Dynamics of Fluids and Plasmas", Proc. of a Symposium, ed. by S. I. PAI, Academic Press, New York and London 1965, p. 239.

We now construct the consolidated expansion of (27) with respect to U-parts. It consists of all irreducible diagrams of (27) with respect to U-parts (i. e. diagrams that do not contain a U-diagram as a part) with all U^0-lines replaced by U-lines (thick wavy lines). Breaking this infinite power series in U at the lowest order and inserting the result into (26) we obtain

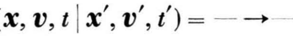

(28)

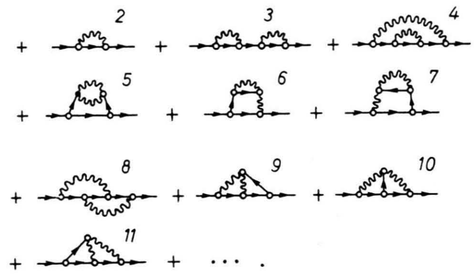
This approximation is equivalent to the quasi-Gaussian assumption (neglecting all cumulants higher than third order), well known from hydrodynamic turbulence theory. It obeys the basic conservation laws, but as it stands, there is no indication that some other realizability conditions, e. g.

$$\langle \tilde{f}(\boldsymbol{x}, \boldsymbol{v}, t)^2 \rangle \geqq 0 ,$$

are satisfied. As is well known the same aproximation applied to incompressible hydrodynamic turbulence indeed does not preserve the positivity of the spectral fluctuation density. However with some additional assumptions such as quasi-homogeneity and quasi-stationarity (which make Fourier transformation useful) and the validity of a dispersion relation for the waves Eq. (28) leads to the kinetic wave equation for $I_k = \langle | \tilde{E}_k |^2 \rangle$ of weak turbulence (see KADOMTSEV [3], Eq. II.51) which guarantees positivity of I_k as can easily be seen. The first term in (28) gives rise to a nonlinear dispersion relation, while the second term represents mode-mode coupling. The main physical deficiency of (28) is the appearance of the elementary propagator G^0. This indicates that the effect of particles being trapped in the potential well of a wave is not included which actually restricts (28) to weakly turbulent systems. Eq. (28) cannot be obtained by Wyld's summation technic; the approximation discussed by Wyld consists of the second term in (25) alone and does not conserve energy.

As has become clear from the discussion of the oscillator problem, a significant improvement over the weak turbulence equation (28) can only be obtained by including an infinite number of terms in the series of which (28) is the lowest order closure. This leads to the introduction of the generalized propagator G. According to Wyld, G consists of all diagrams (parts of U-diagrams) that have two external G^0-lines (one ingoing, the other leaving) and a straight line connecting them:

$G(\boldsymbol{x}, \boldsymbol{v}, t \,|\, \boldsymbol{x}', \boldsymbol{v}', t') =$  (29)

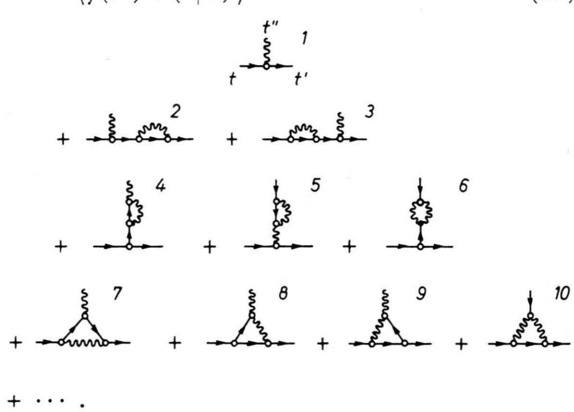


It can be seen by comparing the perturbation expansions that G defined by (29) is identical to the averaged linear response function $\langle \hat{G} \rangle$ introduced by ORSZAG and KRAICHNAN [5] where $\hat{G}$ satisfies the equation

$$L_t \, \hat{G}(t \,|\, t') = \boldsymbol{C} \tilde{f}(t) \cdot \boldsymbol{D} \, \hat{G}(t \,|\, t') + \boldsymbol{D} \tilde{f}(t) \cdot \boldsymbol{C} \, \hat{G}(t \,|\, t') . \tag{30}$$

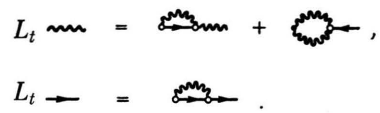
As in the case of the correlation function U we will not carry out summations in the iteration series for G given by (29) but in the diagram expansion of $\langle \tilde{f}(t'') \, \hat{G}(t, t') \rangle$ which appears on the right hand side of (30) after averaging:

$\langle \tilde{f}(t'') \, \hat{G}(t \,|\, t') \rangle =$ (31)



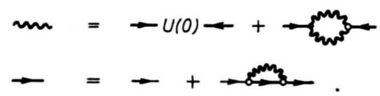
We now construct the consolidated series with respect to U- *and* G-parts by taking all diagrams of (27) and (31) that are irreducible with respect to U- and G-parts [diagrams 1 and 9 – 14 in (27), 1 and 7 – 10 in (31) belong to this class] and replace all U^0-lines by U-lines and all G^0-lines by G-lines (represented by a thick straight line as in section II). Breaking these series at the lowest significant order and substituting the resulting expressions for $\langle \tilde{f} \tilde{f} \tilde{f} \rangle$ and $\langle \tilde{f} \hat{G} \rangle$ into the corresponding

Eqs. (26) and (30) we obtain



(32)

These equations are identical with the symmetric model for the selfconsistent Vlasov problem of Orszag and Kraichnan [5] (the unsymmetric model is obtained by replacing all symmetrized vertices by simple ones). Eqs. (32) obey the basic conservation laws and in adition satisfy a number of realizability conditions as e.g. $\langle \tilde{f}(\boldsymbol{x},\boldsymbol{v},t)^2\rangle \geqq 0$ (see the extensive discussion in the paper of Orszag and Kraichnan). This follows from the fact that Eqs. (32) are the random coupling approximation for the Vlasov equation [analogous to (24) for the random oscillator] and thus have a model representation. Eqs. (32) can also be obtained in Wyld's summation scheme,

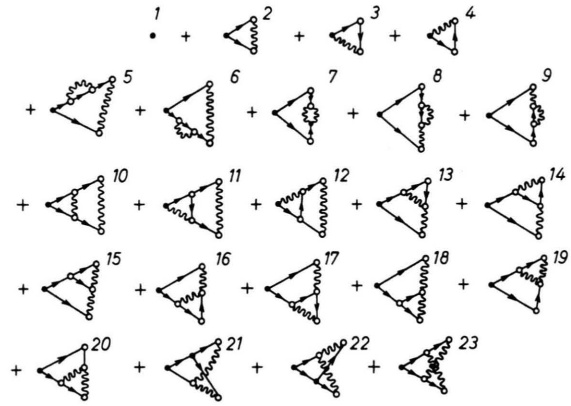


(32 a)

Formally Eqs. (32) are clearly superior to (28). The G-equation in (32) contains the correlation function meaning that in some sense the effect of the random field fluctuations on the particle orbits is taken into account. It is however difficult to see to what extent particle trapping is included in (32), since G has no immediate relation to single particle orbits. To decide this point numerical solutions of (32) for special simple situations are necessary.

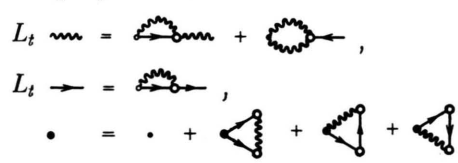
Following the procedure outlined in section II the next approximation requires the introduction of the vertex function Γ. This function is defined as the sum of all diagrams which are parts of U-diagrams and which, like the elementary vertex, have three terminals and a straight line connecting them:

$\Gamma(t\,|\,t',t'') =$ (33)



+ diagrams corresponding to 5 – 23 constructed from 3 and 4 + ...

The consolidated expansions including the vertex function Γ consist of all diagrams of (27), (31) and (33) that are irreducible with respect to U-, G- and Γ-parts [diagrams 1, 12 – 14 in (27); 1, 10 in (31); 2 – 4, 19 – 23 in (33)] and where all U^0 are replaced by U, all G^0 by G and all Γ^0 by Γ. As in section II, Γ is denoted by a solid circle; the symmetrized vertex Γ_s is represented by a thick hollow circle. The lowest order closure leads to the following equations:



(34)

We want to give the algebraic translation of these diagram equations:

$$\left[\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\frac{e}{m}\langle \boldsymbol{E}(\boldsymbol{x},t)\rangle\cdot\frac{\partial}{\partial \boldsymbol{v}}+\frac{\partial F}{\partial \boldsymbol{v}}\cdot\boldsymbol{C}(\boldsymbol{x})\right]U(\boldsymbol{x},\boldsymbol{v},\boldsymbol{x}',\boldsymbol{v}',t')$$
$$=-\frac{\partial}{\partial \boldsymbol{v}}\cdot\int_0^t \mathrm{d}t_1\int \mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{v}_1\,G(\boldsymbol{x},\boldsymbol{v},\mathrm{t}\,|\,\boldsymbol{x}_1,\boldsymbol{v}_1,t_1)\int_0^{t'}\mathrm{d}t_2\int_0^{t'}\mathrm{d}t_3\int \mathrm{d}\boldsymbol{x}_2\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{x}_3\,\mathrm{d}\boldsymbol{v}_3$$
$$\cdot\Gamma_s(\boldsymbol{x}_1,\boldsymbol{v}_1,t_1\,|\,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2,\boldsymbol{x}_3,\boldsymbol{v}_3,t_3)\;\boldsymbol{C}(\boldsymbol{x})\,U(\boldsymbol{x},\boldsymbol{v},t,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2)\,U(\boldsymbol{x}_3,\boldsymbol{v}_3,t_3,\boldsymbol{x}',\boldsymbol{v}',t')$$
$$-\boldsymbol{C}(\boldsymbol{x})\cdot\left[\int_0^t \mathrm{d}t_1\int \mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{v}_1\,G(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}_1,\boldsymbol{v}_1,t_1)\int_0^{t_1}\mathrm{d}t_2\int_0^{t_1}\mathrm{d}t_3\int \mathrm{d}\boldsymbol{x}_2\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{x}_3\,\mathrm{d}\boldsymbol{v}_3\right. \qquad (34\,\mathrm{a})$$
$$\left.\cdot\Gamma_s(\boldsymbol{x}_1,\boldsymbol{v}_1,t_1\,|\,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2,\boldsymbol{x}_3,\boldsymbol{v}_3,t_3)\right]\frac{\partial}{\partial \boldsymbol{v}}\,U(\boldsymbol{x},\boldsymbol{v},t,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2)\,U(\boldsymbol{x}_3,\boldsymbol{v}_3,t_3,\boldsymbol{x}',\boldsymbol{v}',t')$$
$$-\int_0^{t'}\mathrm{d}t_1\int \mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{v}_1 G(\boldsymbol{x}',\boldsymbol{v}',t'\,|\,\boldsymbol{x}_1,\boldsymbol{v}_1,t_1)\int_0^{t_1}\mathrm{d}t_2\int_0^{t_1}\mathrm{d}t_3\int \mathrm{d}\boldsymbol{x}_2\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{x}_3\,\mathrm{d}\boldsymbol{v}_3$$
$$\cdot\Gamma_s(\boldsymbol{x}_1,\boldsymbol{v}_1,t_1\,|\,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2,\boldsymbol{x}_3,\boldsymbol{v}_3,t_3)\,\frac{\partial}{\partial \boldsymbol{v}}\,U(\boldsymbol{x},\boldsymbol{v},t,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2)\cdot\boldsymbol{C}(\boldsymbol{x})\;U(\boldsymbol{x},\boldsymbol{v},t,\boldsymbol{x}_3,\boldsymbol{v}_3,t_3)\,;$$

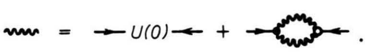
$$\left[\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\frac{e}{m}\langle \boldsymbol{E}(\boldsymbol{x},t)\rangle\cdot\frac{\partial}{\partial \boldsymbol{v}}+\frac{\partial F}{\partial \boldsymbol{v}}\cdot \boldsymbol{C}(\boldsymbol{x})\right] G(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}',\boldsymbol{v}',t')$$
$$-=\frac{\partial}{\partial \boldsymbol{v}}\cdot\int_{t'}^{t}\mathrm{d}t_1\int \mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{v}_1 G(\boldsymbol{x},\boldsymbol{v},\mathrm{t}\,|\,\boldsymbol{x}_1,\boldsymbol{v}_1,t_1)\int_{t'}^{t_1}\mathrm{d}t_2\int_{t'}^{t_1}\mathrm{d}t_3\int \mathrm{d}\boldsymbol{x}_2\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{x}_3\,\mathrm{d}\boldsymbol{v}_3$$
$$\cdot\Gamma_{\mathrm{s}}(\boldsymbol{x}_1,\boldsymbol{v}_1,t_1\,|\,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2,\boldsymbol{x}_3,\boldsymbol{v}_3,t_3)\;\boldsymbol{C}(\boldsymbol{x})\;U(\boldsymbol{x},\boldsymbol{v},t,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2)\;G(\boldsymbol{x}_3,\boldsymbol{v}_3,t_3\,|\,\boldsymbol{x}',\boldsymbol{v}',t')$$
$$-\boldsymbol{C}(\boldsymbol{x})\cdot\left[\int_{t'}^{t}\mathrm{d}t_1\int \mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{v}_1\,G(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}_1,\boldsymbol{v}_1,t_1)\int_{t'}^{t_1}\mathrm{d}t_2\int_{t'}^{t_1}\mathrm{d}t_3\int \mathrm{d}\boldsymbol{x}_2\,\mathrm{d}\boldsymbol{v}_2\,\mathrm{d}\boldsymbol{x}_3\,\mathrm{d}\boldsymbol{v}_3\right. \tag{34 b}$$
$$\left.\cdot\Gamma_{\mathrm{s}}(\boldsymbol{x}_1,\boldsymbol{v}_1,t_1\,|\,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2,\boldsymbol{x}_3,\boldsymbol{v}_3,t_3)\right]\frac{\partial}{\partial \boldsymbol{v}}U(\boldsymbol{x},\boldsymbol{v},t,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2)\,G(\boldsymbol{x}_3,\boldsymbol{v}_3,t_3\,|\,\boldsymbol{x}',\boldsymbol{v}',t'),$$

$$G(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}',\boldsymbol{v}',t')=\delta(\boldsymbol{x}-\boldsymbol{x}')\;\delta(\boldsymbol{v}-\boldsymbol{v}');$$

$$\Gamma(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}',\boldsymbol{v}',t',\boldsymbol{x}'',\boldsymbol{v}'',t'')=\Gamma^0(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}',\boldsymbol{v}',t',\boldsymbol{x}'',\boldsymbol{v}'',t'')$$
$$+\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{v}_1\int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{x}_2\,\boldsymbol{v}_2\,\Gamma(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}_1,\boldsymbol{v}_1,t_1,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2)\int \mathrm{d}t_3\,\mathrm{d}\boldsymbol{x}_3\,\mathrm{d}\boldsymbol{v}_3$$
$$\cdot G(\boldsymbol{x}_1,\boldsymbol{v}_1,t_1\,|\,\boldsymbol{x}_3,\boldsymbol{v}_3,t_3)\int \mathrm{d}t_5\,\mathrm{d}\boldsymbol{x}_5\,\mathrm{d}\boldsymbol{v}_5\Gamma_{\mathrm{s}}(\boldsymbol{x}_3,\boldsymbol{v}_3,t_3\,|\,\boldsymbol{x}_5,\boldsymbol{v}_5,t_5,\boldsymbol{x}',\boldsymbol{v}',t')\int \mathrm{d}t_4\,\mathrm{d}\boldsymbol{x}_4\,\mathrm{d}\boldsymbol{v}_4$$
$$\cdot G(\boldsymbol{x}_2,\boldsymbol{v}_2,t_2\,|\,\boldsymbol{x}_4,\boldsymbol{v}_4,t_4)\int \mathrm{d}t_6\,\mathrm{d}\boldsymbol{x}_6\,\mathrm{d}\boldsymbol{v}_6\Gamma_{\mathrm{s}}(\boldsymbol{x}_4,\boldsymbol{v}_4,t_4\,|\,\boldsymbol{x}_6,\boldsymbol{v}_6,t_6,\boldsymbol{x}'',\boldsymbol{v}'',t'')\int U(\boldsymbol{x}_5,\boldsymbol{v}_5,t_5,\boldsymbol{x}_6,\boldsymbol{v}_6,t_6)$$
$$+\int \mathrm{d}t_1\,\mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{v}_1\int \mathrm{d}t_2\,\mathrm{d}\boldsymbol{x}_2\,\mathrm{d}\boldsymbol{v}_2\Gamma_{\mathrm{s}}(\boldsymbol{x},\boldsymbol{v},t\,|\,\boldsymbol{x}_1,\boldsymbol{v}_1,t_1,\boldsymbol{x}_2,\boldsymbol{v}_2,t_2)\int \mathrm{d}t_3\,\mathrm{d}\boldsymbol{x}_3\,\mathrm{d}\boldsymbol{v}_3 \tag{34 c}$$
$$\cdot G(\boldsymbol{x}_1,\boldsymbol{v}_1,t_1\,|\,\boldsymbol{x}_3,\boldsymbol{v}_3,t_3)\int \mathrm{d}t_4\,\mathrm{d}\boldsymbol{x}_4\,\mathrm{d}\boldsymbol{v}_4\Gamma_{\mathrm{s}}(\boldsymbol{x}_3,\boldsymbol{v}_3,t_3\,|\,\boldsymbol{x}',\boldsymbol{v}',t',\boldsymbol{x}_4,\boldsymbol{v}_4,t_4)\int \mathrm{d}t_5\,\mathrm{d}\boldsymbol{x}_5\,\mathrm{d}\boldsymbol{v}_5$$
$$\cdot G(\boldsymbol{x}_4,\boldsymbol{v}_4,t_4\,|\,\boldsymbol{x}_5,\boldsymbol{v}_5,t_5)\int \mathrm{d}t_6\,\mathrm{d}\boldsymbol{x}_6\,\mathrm{d}\boldsymbol{v}_6\Gamma_{\mathrm{s}}(\boldsymbol{x}_5,\boldsymbol{v}_5,t_5\,|\,\boldsymbol{x}'',\boldsymbol{v}'',t'',\boldsymbol{x}_6,\boldsymbol{v}_6,t_6)\,U(\boldsymbol{x}_2,\boldsymbol{v}_2,t_2,\boldsymbol{x}_6,\boldsymbol{v}_6,t_6),$$

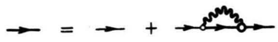
where Γ^0 is given by (11) and $\Gamma_{\mathrm{s}}(X\,|\,X'\,X'')=\Gamma(X\,|\,X',X'')+\Gamma(X\,|\,X'',X')$.

These equations which have not been derived before, are the analogue of the vertex approximation (25) for the oscillator problem. Like the approximations (28) and (32) they are consistent with a number of conservation laws (e. g. particle number-, momentum-, and energy conservation) which can easily be verified from (34). It is interesting to note that the validity of these conservation laws only depend on the form of Eq. (34 a) for U [and Eq. (3) for F] and is independent of the specific values of the functions G and Γ. The crucial point is that the first vertex in the terms on the right hand side of the Eqs. (34) is an elementary vertex Γ^0 which comes out naturally in our derivation. A similar approximation in Wyld's summation scheme would entail a generalized vertex at this place, just because of the symmetry of $U(X,X')$:

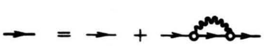


(35)

The equation for G could for instance be chosen as in (34 b)


(36)

which, however, would leave out diagrams like number 11 in (29) without any argument in Wyld's theory. Or one could take


(37)

as was done by Shutko [12] in hydrodynamic turbulence. This choice however would entail apparent double counting of diagrams. Or one could add higher order irreducible diagrams to the right hand side of (36). None of these attempts leads to an energy conserving equation for U when inserted into Eq. (35).

On the other hand we have no idea how to derive (34) by Kraichnan's diagram technic [7], applied to a nonlinear problem like the self-consistent Vlasov equation. Kraichnan's diagrams which have a certain resemblance to those of Wyld in the linear case (e. g. the oscillator problem) become rather obscure in the nonlinear case.

As in the oscillator problem we cannot prove that equations (34) satisfy the statistical realizability inequalities as $\langle \tilde{f}(\boldsymbol{x},\boldsymbol{v},t)^2\rangle \geqq 0$. Nevertheless the success of the vertex approximation for the random oscillator suggests that (34) is a realizable approximation and a significant improvement over the random coupling equations (32).

IV. Conclusions

We have derived and discussed three different statistical approximations to the Vlasov equation

[12] A. V. Shutko, Soviet Phys.-Doklady 9, 857 [1965].

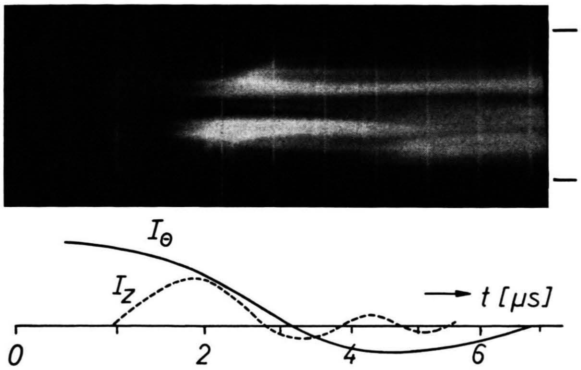


Abb. 9. $Z-\Theta$-Entladung mit Feldüberlagerung.

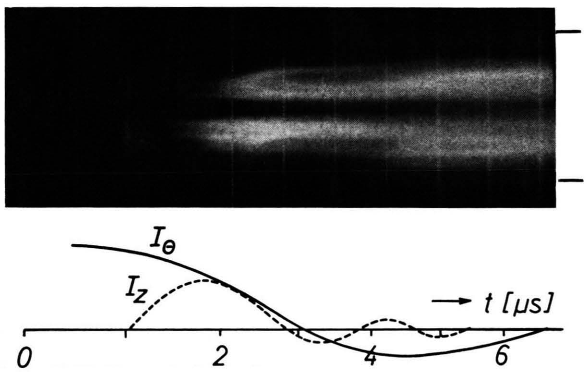


Abb. 10. $Z-\Theta$-Entladung mit Feldüberlagerung.

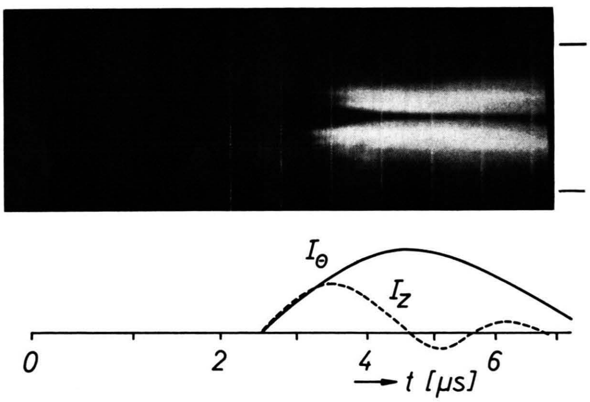


Abb. 11. $Z-\Theta$-Entladung mit Feldüberlagerung.

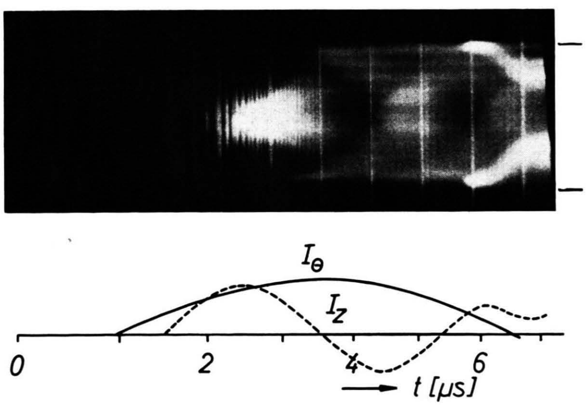


Abb. 12. $Z-\Theta$-Entladung ohne Feldüberlagerung.

hopefully giving an increasingly better description of collisionless plasma turbulence. By requiring analogy to the linear oscillator problem and consistency with the dynamical conservation laws these approximations are uniquely determined. Some critical remarks should however be added. Though intuitively satisfactory and numerically confirmed in the simple oscillator problem there is of course no proof that the selection of diagrams described above should necessarily lead to physically meaningful asymptotic equations. Equations (32) inspire some additional confidence, since they have a model representation (for an extensive discussion of the consequences see the paper of Orszag and Kraichnan) [13]. This could however be accidental which is suggested by the fact that the vertex approximation of the oscillator problem Eq. (25) being clearly superior to the "random coupling approximation" Eq. (24) seems to have no model representation.

We have not considered the Eq. (3) for F which differs in the different approximations only by the specific expression for $\langle \tilde{E} \tilde{f} \rangle$. It seems to be rather difficult to find out whether these approximations imply a physically meaningful behavior of F, i. e. whether F remains positive if it is so initially or at most develop only slightly negative parts of small weight for the velocity space moments. In the simplest approximation, neglecting all nonlinearities in the equation for U, i. e. $U = U^0$, (3) is just the equation of quasi-linear theory. At least in the usual form of a diffusion equation this equation preserves positivity of F and does not lead to an instability for an initial F with a slightly negative part, which is a consequence of the positive diffusion coefficient. In higher approximations, such as discussed in this paper, this need no longer be the case. A negative diffusion coefficient can decrease the distribution function in some region to more and more negative values which would imply steeper and steeper positive gradients which feed the instability. This point must be clarified by numerical computations.

All the approximations discussed have to be corrected to distinguish between the "resonant" interaction of modes with neighboring wavelengths and the "adiabatic" interaction between modes of very different length scales (see KADOMTSEV [3], chapter III, KRAICHNAN [14]). This could be done for instance by introducing a Lagrangian description (as was done by KRAICHNAN [15] for hydrodynamic turbulence) which would essentially amount to performing time-integrations in the equations not at a fixed phase-space point but along the actual phase-space orbit of a particle.

[13] Besides, it is the only approximation that can be obtained by Wyld's summation method.

Acknowledgements

The author is indebted to Dr. D. PFIRSCH and Dr. P. GRÄFF for a number of fruitful discussions of this work.

Appendix

Here we will prove that there are no diagrams in the expansion of U with closed-tree like parts. Consider all diagrams of n-th order in the expansion of $\tilde{f}(\boldsymbol{x}, \boldsymbol{v}, t)$, Eq. (14), that have the same skeleton (see Fig. 2), i. e. the same G^0-line structure (not using symmetrized vertices). We call this sum $A^{(n)}$.

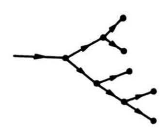


Fig. 2. A 9th order skeleton.

$A^{(n)}$ has the form: $A^{(n)} = B^{(n)} - \langle B^{(n)} \rangle$, where $B^{(n)}$ consists of all those diagrams, where a diagram of a special $A^{(m_1)}$ is connected with a diagram of a special $A^{(m_2)}$ in the first vertex from the left, $m_1 + m_2 = n - 1$ (in Fig. 2: $m_1 = 3$, $m_2 = 5$) and $\langle B^{(n)} \rangle$ consists of all possible total contractions of $B^{(n)}$ (all possible closed-tree like diagrams of $B^{(n)}$). $B^{(n)}$ may be written

$$B^{(n)} = A^{(m_1)} \times A^{(m_2)}. \tag{38}$$

Now the $A^{(m_1)}$, $A^{(m_2)}$ again have the form

$$A^{(m_1)} = B^{(m_1)} - \langle B^{(m_1)} \rangle,$$
$$A^{(m_2)} = B^{(m_2)} - \langle B^{(m_2)} \rangle;$$
$$B^{(m_1)} = A^{(l_1)} \times A^{(l_2)},$$
$$B^{(m_2)} = A^{(l_3)} \times A^{(l_4)},$$
$$l_1 + l_2 = m_1 - 1, \qquad l_3 + l_4 = m_2 - 1.$$

[14] R. H. KRAICHNAN, Phys. Fluids **7**, 1723 [1964].
[15] R. H. KRAICHNAN, Phys. Fluids **8**, 575 [1965].

Thus we have

$$\begin{aligned} A^{(n)} &= B^{(n)} - \langle B^{(n)} \rangle \\ &= (B^{(m_1)} - \langle B^{(m_1)} \rangle) \times (B^{(m_2)} - \langle B^{(m_2)} \rangle) \\ &\quad - \langle B^{(n)} \rangle = [\,(B^{(l_1)} - \langle B^{(l_1)} \rangle) \times (B^{(l_2)} - \langle B^{(l_2)} \rangle) \\ &\quad - \langle B^{(m_1)} \rangle] \times [\,(B^{(l_3)} - \langle B^{(l_3)} \rangle) \\ &\quad \times (B^{(l_4)} - \langle B^{(l_{42})} \rangle) - \langle B^{(m_2)} \rangle] - \langle B^{(n)} \rangle \end{aligned} \tag{39}$$

etc.

We now investigate the average of $A^{(n)}$, in the form (39), with an arbitrary diagram D of $\tilde{f}$ say of k-th order ($n+k$ should be even, otherwise the average will vanish identically). Because of the definition of $\langle B^{(n)} \rangle$ and the first line of (39) there will be no total contractions of $A^{(n)}$-diagrams in the average. We now consider those contractions where the $B^{(m_2)}$-parts are connected in some way with D while the $B^{(m_1)}$-parts are totally contracted in all possible ways, and similar contractions with $B^{(m_1)}$ and $B^{(m_2)}$ interchanged. Because of the factors $B^{(m_1)} - \langle B^{(m_1)} \rangle$ resp. $B^{(m_2)} - \langle B^{(m_2)} \rangle$ the sum of all diagrams with $B^{(m_1)}$- or $B^{(m_1)}$-contractions will also vanish. In the same way, considering contractions which contain all possible total contractions of some $B^{(l)}$ it is evident from the third line of (39) that there will be no diagrams with closed-tree like $B^{(l)}$-parts. By going to smaller and smaller subparts it is clear that no closed-tree like parts at all will appear in $A^{(n)}$ in the average of $A^{(n)}$ and D. Because U is symmetric to interchange of right and left parts of all diagrams, it follows immediately that U will not contain any diagrams with closed-tree-like parts.

By the same arguments it can be shown that there are no closed-tree like parts in the diagrams of $\langle \tilde{f}\tilde{f}\tilde{f} \rangle$, Eq. (27) and $\langle \tilde{f}\hat{G} \rangle$, Eq. (31).